Journal of Statistical Physics, Vol. 50, Nos. 1/2, 1988

Noise and Bifurcations

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Received April 1, 1987; revision received August 25, 1987

The influence of while noise on bifurcating dynamical systems is investigated using both Fokker–Planck and functional integral methods. Noise leads to fuzzy bifurcations where physically relevant quantities become smooth functions of the bifurcation parameters. We study dynamical and probabilistic quantities, such as invariant measures, Liapunov exponents, correlation functions, and exit times. The behavior of these quantities near the deterministic bifurcation point changes for distinct values of the control parameter. Therefore the very concept of bifurcation point becomes meaningless and must be replaced by the notion of bifurcation region.

KEY WORDS: Dynamical systems; bifurcations; noise; invariant measures; Liapunov exponent.

1. INTRODUCTION

The time evolution of physical systems with a finite number of degrees of freedom may be described by simple dynamical systems, namely systems of ordinary differential equations. Such situations occur, for instance, in Hamiltonian or dissipative classical mechanics and homogeneous chemical kinetics (evolution of species concentrations in a well-stirred tank).

Such dynamical systems generally depend on several control parameters. When those parameters are varied, the topological properties of the system may change, a phenomenon known as bifurcation. For instance, the nature of the attractors that describe the asymptotic behavior of solutions may change from periodic (stable fixed points or limit cycles) to nonperiodic (so-called strange attractors); the system then displays a transition from order to chaos.

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0022-4715/88/0100-0345$06.00/0

Generally such simple macroscopic systems do not account for the full dynamics, as they neglect, at least partly, microscopic degrees of freedom. The influence of such small-scale processes can be modeled by adding a relevant noise term to the original system. The first historical example of such an approach is the well-known Brownian motion problem, which leads to the Langevin equation.

Topological properties are no longer relevant when noise is taken into account and should be replaced by measure-theoretic (invariant measure), dynamical (Liapunov exponent), or more probabilistic ones (exit time from a specified domain, for instance). To enlighten that point, let us examine a simple system which posesses a globally and linearly stable fixed point (this example will be revisited later). When a small-amplitude Gaussian white noise is added, the asymptotic behavior is no longer described by that fixed point (which is a topological feature); for almost every sample path the whole space is explored in the asymptotic regime with a frequency given by an absolutely continuous ergodic invariant measure (which weakly converges to a Dirac measure when the noise goes to zero). The associated density shows a strong enhancement near the deterministic fixed point. One may further investigate the effects of noise by calculating the Liapunov exponent (which quantifies the system's global sensitivity to a change in the initial condition), and correlation or response functions. One may also examine local properties, such as the mean exit time from a given neighborhood of the deterministic fixed point.

We expect the sensitivity of a deterministic system to noise to depend greatly on its intrinsic stability. When the system is structurally stable, i.e., when small perturbations lead to a conjugate system exhibiting the same topological properties, the effects of noise are generally small and essentially appear as a blurring of the system's features (as will be shown in Section 3). More drastic changes are expected near a bifurcation point in parameter space, since the system is then sensitive to arbitrary small perturbations.

In what follows, we inquire into the effects of noise on dynamical systems in the vicinity of a bifurcation point. The noisy system is obtained via the addition of a noise term to the deterministic dynamical system, which yields a Langevin-type equation.[1] Many results have already been obtained by studying the associated Fokker–Planck equation,[2] which makes it possible in particular to shed some light on noise-induced transitions.[3] The concept of center manifold has been extended to noisy systems, which justifies the very use of the Langevin equation.[4–7] The effect of noise has also been considered from the viewpoint of nonequilibrium thermodynamics.[8] In the present work we focus on the following point: Is it possible to give an alternative nontopological definition of bifurcations, which remains meaningful for a noisy system? By examining simple systems (bifurcation normal forms), we show that the sharp deterministic bifurcation is replaced by a smooth transition zone. Therefore the answer to our question is in the negative.

We emphasize the point that since we are dealing with one-dimensional systems (time is the only dimension in the problem), the only changes of behavior that may be assimilated to phase transitions occur at "zero temperature," that is, in the deterministic case (since the variance of noise σ^2 plays the part of temperature). They correspond to deterministic bifurcations. The reader must therefore beware of fallacious analogies.

The remainder of the paper is divided into six sections: We first introduce the required formalism (Section 2): Langevin equation, functional integral, Foker–Planck equation, relevant quantities. In Section 3 very simple systems are investigated, casting some light on the basic effects of noise. Section 4 recalls some fundamentals of bifurcation theory (codimension, normal forms) and provides a frame for the problems discussed in the following sections.

Section 5 discusses the behavior of the invariant measure for noisy systems near a deterministic bifurcation point. Both saddle-node and pitchfork bifurcations are investigated. We show that the shape of the invariant density changes for the deterministic bifurcation value or undergoes no change, depending on the bifurcation one considers. Therefore this quantity seems rather inappropriate to investigate nontrivial efects of noise on a bifurcating system. In Section 6 we calculate (by analytical or numerical means) the Liapunov exponent for the systems previously introduced. We show that the addition of noise alters the sensitivity to changes in the initial condition (stabilization of the system, noise-dependent shift of the point where the Liapunov exponent vanishes or reaches an extremum). In Section 7 we study the effective potential for the supercritical pitchfork bifurcation. We show that an extended transition zone replaces the deterministic bifurcation. Conclusions of the present study are drawn in Section 8.

Sections 2–4, which have been included to keep the present study as self-contained and elementary as possible, may be skipped by the reader already familiar with basic effects of noise on dynamical systems and bifurcation theory.

2. FORMALISM

This part is devoted to the formalism required for dealing with noisy dynamical systems.[9–12] For the sake of simplicity, we consider one-dimensional systems driven by an additive Gaussian noise $b(t)$. The evolution of such a system is described by a stochastic differential equation, known as the Langevin equation, which reads

$$dx/dt = f_\varepsilon(x) + b(t), \qquad x(0) = X_0 \tag{2.1}$$

$f_\varepsilon(x)$ is the deterministic part, which we assume to depend on the sole parameter ε, and X_0 denotes the initial condition. The zero-mean Gaussian process $b(t)$ is completely characterized by its covariance function $\mathrm{Cov}(t, t')$. For a white noise [$b(t)\,dt = \sigma\,dw(t)$, with $w(t)$ the standard Wiener process] that function reads

$$\mathrm{Cov}(t, t') = \sigma^2 \delta(t - t') \tag{2.2}$$

The characteristic functional $C[y]$ of a stochastic process defined on the interval $[0, T]$ may be deduced from the covariance function

$$C[y] = \exp\left[-\frac{1}{2}\int_0^T dt\,dt'\; y(t)\,\mathrm{Cov}(t, t')\,y(t')\right] \tag{2.3}$$

For a Gaussian white noise it reduces to

$$C[y] = \exp\left[-\tfrac{1}{2}\sigma^2 \int_0^T dt\; y(t)^2\right] \tag{2.4}$$

This functional completely determines, via its Fourier transform, the probability distribution of noise

$$P[b] = \det{}^{-1}(\partial_t) \int \mathscr{D}\left[\frac{y}{2\pi}\right] C[y] \exp\left(i \int_0^T dt\; yb\right) \tag{2.5}$$

The determinant and the factor 2π in the integration measure of this functional integral ensure the correct normalization of $P[b]$.

Henceforth, we shall restrict ourselves to the simple case of Gaussian white noise.

Let us introduce the transition probability $P(X, T \mid X_0)$, which is the measure of the set of path that start at X_0 and reach X at time T. If $x_b(T \mid X_0)$ denotes the value at time T of the solution of (2.1) for a given sample path $b(t)$ of noise, $P(X, T \mid X_0)$ may be expressed as the average of $\delta[X - x_b(T \mid X_0)]$ over the noise probability density (2.5),

$$P(X, T \mid X_0) = \langle \delta[X - x_b(T \mid X_0)] \rangle$$

Using (2.5), we have

$$\langle \delta[X - x_b(T \mid X_0)] \rangle = \det{}^{-1}(\partial_t) \int \mathscr{D}\left[\frac{y}{2\pi}\right] \mathscr{D}b\, \mathscr{D}x \\ \times \delta[x - x_b(t \mid X_0)] \left[\exp\left(i \int_0^T dt\; yb\right)\right] C[y] \tag{2.6}$$

The dependence on noise of the delta functional may be explicitly written

$$\delta[x - x_b(t \mid X_0)] = \det |\delta b/\delta x| \; \delta[\dot{x} - f_\varepsilon(x) - b(t)] \; \delta[x(0) - X_0] \tag{2.7}$$

The Jacobian term is readily evaluated as[13]

$$\det \left| \frac{\delta b}{\delta x} \right| = \det(\partial_t) \exp\left(-\frac{1}{2} \int_0^T dt \frac{df}{dx} \right) \tag{2.8}$$

by using the Stratonovich stochastic integral and forward propagation in time. Inserting (2.4), (2.7), and (2.8) into (2.6) and performing the integration over b, we finally obtain

$$P(X, T \mid X_0) = \int_{X_0}^{X} \mathscr{D}\left[\frac{y}{2\pi}\right] \mathscr{D}x \exp\left[-\frac{\sigma^2}{2} \int_0^T dt\, y^2 + i \int_0^T dt\, y(\dot{x} - f) - \frac{1}{2} \int_0^T dt \frac{df}{dx} \right] \tag{2.9}$$

where the initial and final conditions imposed by the delta functions appear as integration bounds.

We now derive the Fokker–Planck equation with the help of the Feynman–Kac formula.[14] Integrating over y in (2.9) (a straightforward Gaussian integration) yields

$$P(X, T \mid X_0) = \det{}^{1/2}(\partial_t) \int_{X_0}^{X} \mathscr{D}\left[\frac{x}{(2\pi)^{1/2}}\right] \times \exp\left\{ -\frac{1}{\sigma^2} \int_0^T dt \left[(\dot{x} - f)^2 + \sigma^2 \frac{df}{dx} \right] \right\} \tag{2.10}$$

We integrate by parts the argument of the exponential, which yields

$$P(X, T \mid X_0) = \exp\left[\frac{U(X) - U(X_0)}{\sigma^2} \right] \times \int_{X_0}^{X} \mathscr{D}w[x] \exp\left[-\frac{1}{\sigma^2} \int_0^T dt\, V(x) \right] \tag{2.11}$$

where

$$U(X) = \int dx\, f_\varepsilon(x), \qquad V(x) = \frac{1}{2}\left[\left(\frac{dU}{dx}\right)^2 + \sigma^2 \frac{d^2U}{dx^2} \right] \tag{2.12}$$

and functional integration is performed using the Wiener measure $\mathscr{D}w$. The path integral in (2.11) is formally equivalent to the propagator kernel of quantum mechanics, and the Feynman–Kac formula enables us to rewrite (2.11) as

$$P(X, T \mid X_0) = \exp\left[\frac{U(X) - U(X_0)}{\sigma^2}\right] \sum_{n=0}^{\infty} [\exp(-\lambda_n t)]\, \psi_n(X)\, \psi_n(X_0) \quad (2.13)$$

where $\psi(x)$ satisfies the Schrödinger equation

$$-\frac{\sigma^4}{2}\frac{d^2\psi}{dx^2} + V\psi = \lambda\psi \quad (2.14)$$

Therefore, the transition probability is the solution of the Fokker–Planck (FP) equation

$$\partial_t P + \partial_x(fP) = \tfrac{1}{2}\sigma^2\, \partial_{xx} P \quad (2.15)$$

with initial condition $P(x, 0 \mid X_0) = \delta(x - X_0)$ (i.e., the so-called principal solution of the FP equation).

If we had used backward propagation in time in (2.8), the sign of the exponential in the Jacobian would have changed, yielding an effective potential of the form

$$\bar{V}(x) = \frac{1}{2}\left[\left(\frac{dU}{dx}\right)^2 - \sigma^2\frac{d^2U}{dx^2}\right] \quad (2.16)$$

This potential is related to the Kolmogorov equation, the adjoint equation of (2.15),

$$\partial_t W - f\,\partial_{X_0} W = \tfrac{1}{2}\sigma^2\,\partial_{X_0X_0} W, \qquad W(X_0, 0 \mid X) = \delta(X - X_0) \quad (2.17)$$

where W is the transition probability considered as a function of the *initial* condition X_0.

The system's invariant measure $\mu(X)\,dX$, which is the stationary solution of the Fokker–Planck equation, is also given by the long-time behavior of $P(X, T \mid X_0)$,

$$\lim_{T \to \infty} P(X, T \mid X_0) = \mu(X) \quad (2.18)$$

From (2.13) it is easily seen that this limit exists if the Schrödinger equation posesses a normalized eigenfunction ψ_0 associated with the eigenvalue $\lambda = 0$ (which requires some constraints on f_ε). Associated with this eigenvalue is a solution of the form $\psi_0 = \exp(S/\sigma^2)$, where S satisfies the equation $-\sigma^2 S''/2 - (S')^2/2 + V = 0$. Therefore, $S = U$ and the density of the invariant measure reads

$$\mu(X) = \mathcal{N} \exp\left(\frac{2}{\sigma^2}U\right) \quad (2.19)$$

Notice that, due to the special form of the effective potential V, the solution $S=U$ is obtained *irrespective* of the value of σ. Hence, the WKB approximation becomes exact in the limit $T\to\infty$. The equation satisfied by S for small noise, $-(S')^2/2+V_0=0$, $V_0=f^2/2$, is similar to the Hamilton–Jacobi equation, for the zero energy level, associated with the Hamiltonian $H=[(dx/dt)^2-f^2]/2$. Therefore, in order to derive the invariant measure from the functional integral (2.10), it suffices to compute the minimum of the exponential term inside the integral, neglecting terms in σ^2. This minimum, which corresponds to a trajectory with zero energy, is just the action $S=U$.

The invariant probability measure $\mu(X)\,dX$ is ergodic,[15]

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\, f(x(t))=\int_{-\infty}^{+\infty}dx\,\mu(x)\,f(x) \tag{2.20}$$

This allows us to compute the Liapunov exponent λ of the noisy dynamical system (2.1),

$$\lambda=\lim_{T\to\infty}\ln|\delta x|=\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,\frac{df}{dx}=\int_{-\infty}^{+\infty}dx\,\mu(x)\frac{df}{dx} \tag{2.21}$$

[$\delta x(t)$ is the separation between two given trajectories, the initial conditions of which differ by δX_0].

Another quantity of interest is the exit time from a domain,[16] which measures the "lifetime" of an attractor of the noise-driven dynamical system. The exit time τ_D from an interval $D=(a,b)$, starting from the initial condition X_0, can be defined as

$$\tau_D(X_0)=\int_D dx\int_0^\infty dt\;W(X_0,T\,|\,x) \tag{2.22}$$

Using the Kolmogorov backward equation (2.17), one obtains

$$f_\varepsilon(X_0)\,\partial_{X_0}\tau_D+\tfrac{1}{2}\sigma^2\,\partial_{X_0X_0}\tau_D=-1 \tag{2.23}$$

with appropriate boundary conditions. For example, in the problem of exit through a potential barrier one imposes reflection at one boundary, $\tau'_D(a)=0$, and absorption at the other, $\tau_D(b)=0$, where a and b are located on opposite sides of the barrier. The solution of (2.23) then reads

$$\tau_D(X_0)=\frac{2}{\sigma^2}\int_{X_0}^b dx\int_a^x dy\,\frac{\mu(y)}{\mu(x)} \tag{2.24}$$

The long-time behavior of the stochastic system for small noise can be studied by evaluating via the Laplace method the functional integral (2.10), which expresses the transition probability.[17] The dominant contribution is given by the minimum of the so-called action functional $S_\sigma[x]$,[18]

$$S_\sigma[x]=\frac{1}{2}\int_0^T dt\left\{[\dot{x}-f_\varepsilon(x)]^2+\sigma^2\frac{df_\varepsilon}{dx}\right\} \tag{2.25}$$

The extremal trajectories $x_c(t)$ are solutions of

$$\ddot{x}_c=\frac{dV_0}{dx_c}, \qquad V_0=\frac{1}{2}f_\varepsilon^2(x_c) \tag{2.26}$$

Such an extremal trajectory is a path that maximizes the transition probability between the given endpoints; generally, it is *not* a solution of the deterministic system.

The asymptotic form of the invariant measure can be derived in this framework,

$$-\lim_{\sigma\to 0}\sigma^2\ln\mu(x)=\operatorname{Inf}S_0[x] \tag{2.27}$$

the infimum being taken in the set of trajectories that are defined on the semi-infinite interval $[0,+\infty[$ and leave the deterministic stable fixed points. Therefore, $\mu(X)$ builds up from large-deviation events with exponentially small probabilities.

Dynamical quantities, such as correlation functions, can be investigated via the generating functional, which is defined by[19,20]

$$Z[J]=\frac{1}{Z}\int\mathscr{D}x\exp\left(-\frac{S_\sigma[x]}{\sigma^2}\right)\exp\left(i\int Jx\,dt\right), \qquad Z[0]=1 \tag{2.28}$$

For instance, the two-point correlation function is given by

$$G(t,t')=-\frac{\delta^2 Z[J]}{\delta J(t)\,\delta J(t')} \tag{2.29}$$

Response functions can similarly be derived from a generating functional. As is well known, connected Green's functions are generated by the functional $W[J]=-i\ln Z[J]$, whereas the so-called one-particle irreducible functions are related to $\Gamma(\phi)$, the Legendre transform of $W[J]$,

$$\Gamma[\phi]+W[J]=\int dt\,J\phi, \qquad \phi=\delta W/\delta J \tag{2.30}$$

Usually one expands $\Gamma(\phi)$ in powers of the derivatives of ϕ around a constant trajectory (since the main contribution to $\Gamma[\phi]$ comes from such trajectories). The first term (which does not depend on the derivatives) then reads

$$\Gamma_0[\phi] = \int V_{\text{eff}}[\phi]\, dt \tag{2.31}$$

where the function V_{eff} is known as the *effective potential.* It enables us to obtain the vertex functions for zero external "momenta" and to define "renormalized" coefficients in the action.

The functional integral formulation of stochastic processes of the form (2.1) admits an immediate generalization to *d*-dimensional or infinite-dimensional processes.[21] The Fokker–Planck formalism may also be adapted to such cases, since the stochastic processes remain Markovian.

On the other hand, general Gaussian correlated processes (2.3), as is well known, cannot be treated, at least exactly, in the framework of the Fokker–Planck equation, and approximate schemes must be developed.[22] Such a problem does not arise in the framework of functional integration, where it suffices to replace the local action in (2.9) by a nonlocal one using (2.3).[23] Obviously, the Feynman–Kac formula is no longer valid in this case and the relation between the functional integral and a partial differential equation breaks down.

3. BASIC EFFECTS OF NOISE

In this section we illustrate some basic effects of noise.[24] First we revisit the example of a linear system given in Section 1,

$$dx/dt = \varepsilon x, \qquad \varepsilon \neq 0 \tag{3.1}$$

When the fixed point $x = 0$ is globally stable ($\varepsilon < 0$), the asymptotic behavior is described by a Dirac measure located at $x = 0$.

If we add a Gaussian white noise with standard deviation σ, the invariant measure becomes absolutely continuous with Gaussian density (stationary solution of the Fokker–Planck equation)

$$\mu(x) = (1/N)\exp(-\varepsilon x^2/\sigma^2) \tag{3.2}$$

The system spends most of the time in a neighborhood of the deterministic fixed point, the size of which, $\sigma(2\varepsilon)^{-1/2}$, increases with the dimensionless parameter $\beta = \varepsilon^{-1/2}\sigma$; this parameter quantifies the balance between the conflicting effects of deterministic stability and noise-induced diffusion.

Noise thus blurs the deterministic system's features. However, the effects of noise are deeper than is suggested by the behavior of the density around $x=0$. Indeed, the trajectories no longer converge to the stable fixed point. Due to large fluctuations, they wander erratically and visit distant regions often enough to rule out any vanishing of the density. Therefore, the system's topological properties are completely destroyed by noise: instead of a stable fixed point (topological concept), we are faced with an invariant measure (probabilistic quantity) with the whole space as support and presenting a local enhancement at $x=0$.

When $\varepsilon>0$ the system's instability forbids the existence of any ergodic invariant measure even in the presence of noise.

In the deterministic case the Liapunov exponent (which measures the dynamical stability) is the derivative of the vector field at the fixed point. In the presence of noise it remains a meaningful quantity and retains its deterministic value $\lambda=\varepsilon$, as can be calculated from its very definition (time average along a trajectory). Thus, noise does not affect the sensitivity to the initial condition. This results from the deterministic system's linearity, as emphasized by the following example.

Let us add a (stabilizing) cubic term to a stable linear vector field

$$dx/dt=\varepsilon x+\kappa x^3+\sigma b(t), \qquad \varepsilon<0, \quad \kappa<0 \tag{3.3}$$

In the deterministic case the asymptotic behavior is governed by the sole linear part; the Liapunov exponent remains equal to ε. In the presence of noise the density of the invariant measure reads

$$\mu(x)=\frac{1}{N'}\exp\left[\frac{1}{\sigma^2}\left(\frac{\kappa x^4}{2}+\varepsilon x^2\right)\right] \tag{3.4}$$

and the Liapunov exponent

$$\lambda=\varepsilon-3\kappa\int_{-\infty}^{+\infty}x^2\mu(x)\,dx \tag{3.5}$$

A stabilization of the system occurs (the Liapunov exponent decreases with respect to the linear case) together with a localization of the invariant measure around $x=0$ (the tail of the density decreases more rapidly than before). Thus, noise brings the nonlinear terms into play. The explanation is obvious; large fluctuations send trajectories far from the deterministic fixed point in regions where the nonlinearity dominates the deterministic dynamics.

In this section we have dealt only with structurally stable systems (i.e., insensitive to perturbations) and, as expected, we observed quantitative rather than qualitative effects of noise on the nontopological properties. In what follows we consider the opposite case of structural instability, that is, the bifurcation problem.

4. BIFURCATIONS

We shall now proceed to an informal review of some fundamentals of bifurcation theory.[25,26] Consider the space V of C^{∞} vector fields on $\mathbb{R}^n$ with an equilibrium position at the origin of coordinates. Let F_{ε} be a one-parameter family in V. For most values of the parameter ε the vector field is structurally stable. This stability will eventually break down for certain values of the control parameter known as bifurcation values.

Among bifurcations involving the fixed point, one distinguishes between local bifurcations, which are amenable to a local study near the equilibrium (change of stability of a fixed point, for instance), and global ones, such as homoclinization[27,28] (connection of the stable and unstable manifolds far from the fixed point).

The dimensionality of the locus of bifurcation points in V is called the codimension of the bifurcation. The genericity and stability of a bifurcation depend on this codimension.

A codimension-one local bifurcation occurs on a hypersurface of V and is generic (genericity of the transversal crossing of curves and hypersurfaces). In addition, families of vector fields nearby a bifurcating family will undergo the same bifurcation. This entails the stability of codimension-one bifurcations. The two such bifurcations are the saddle-node bifurcation and the Hopf (subcritical or supercritical) bifurcation.

Higher codimension bifurcations occur for vector fields on a surface that is not crossed by a generic one-parameter family. They are unstable, since families of vector fields lying arbitrarily near a bifurcating family may be found that do not display the same bifurcation scheme.

If we limit ourselves to a subspace of V, the codimension of a given bifurcation may change. For instance, the pitchfork and transcritical bifurcations become generic if the vector fields exhibit certain symmetry properties.

To study a local bifurcation, one need only to consider its normal form. Relevant phenomena are limited to a local submanifold of $\mathbb{R}^n$, known as the local center manifold. Therefore, one may locally change variables so that the dynamics on the center manifold is described by a simple differential system, the so-called normal form, which embodies the whole complexity of the bifurcation.

In what follows we investigate the effects of noise on simple bifurcations: the pitchfork and saddle-node bifurcations. In spite of its mathematical and physical interest, the Hopf bifurcation[2,5,7,8,29] will not be dealt with in this paper, which is limited to bifurcations with one-dimensional center manifolds.

The normal form of pitchfork bifurcation reads

$$\dot{x} = \varepsilon x \pm x^3 \tag{4.1}$$

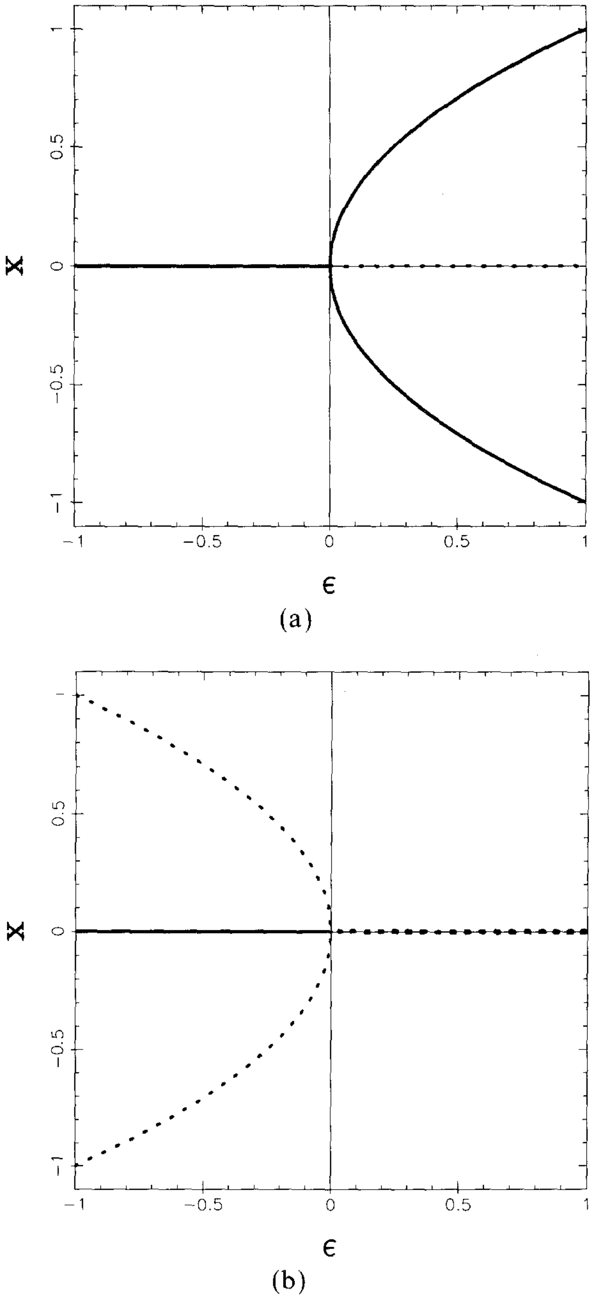

Fig. 1. Bifurcation diagrams for pitchfork bifurcations displaying (—) stable fixed points together with (···) unstable ones. (a) Supercritical case. (b) Subcritical case.

This bifurcation is generic if we restrict ourselves to a subset of V: the vector fields whose first and second derivatives along the center manifold vanish at the fixed point. According to the sign of the third derivative, the bifurcation is either supercritical (minus sign in the normal form) or subcritical (plus sign).

In the supercritical case, a stable fixed point ($\varepsilon < 0$) changes stability by giving birth to a pair of stable fixed points ($\varepsilon > 0$) (see Fig. 1a). In the subcritical case, a pair of unstable fixed points exists for $\varepsilon < 0$ (see Fig. 1b).

The normal form of saddle-node bifurcation reads

$$\dot{x} = \varepsilon + x^2 \tag{4.2}$$

This bifurcation is generic in V and corresponds to the collapse at $\varepsilon = 0$ of two fixed points with opposite stability (see Fig. 2). It plays an important part in type 1 intermittency,[30] a continuous transition from order to chaos. This transition requires a local saddle-node bifurcation together with a reinjection process, which bring trajectories back to the region where the bifurcation takes place. That reinjection may be ensured by a homoclinic trajectory.

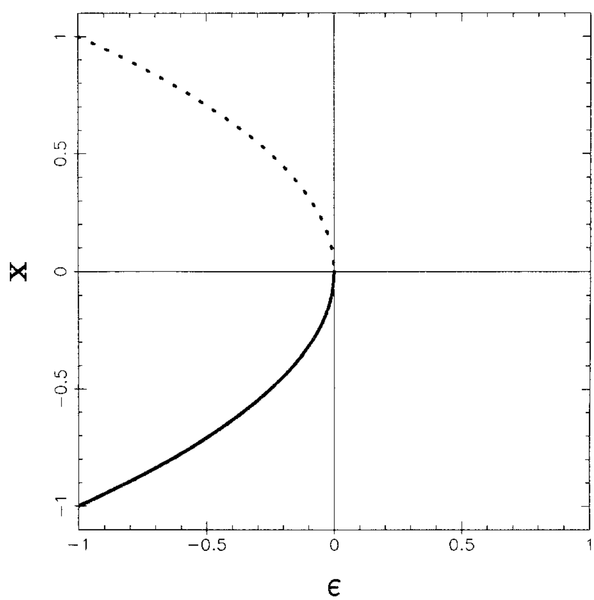

Fig. 2. Bifurcation diagram for saddle-node bifurcation displaying (——) the stable fixed point and (···) the unstable one.

5. INVARIANT MEASURE

For a noise-driven dynamical system, bifurcations can no longer be defined as a change in the topological properties. However, in the deterministic case changes in nontopological quantities also occur at the bifurcation point. The invariant measure, for instance, is different on either side of the saddle-node or pitchfork bifurcation; the Liapunov exponent changes from negative (stable fixed point) to positive (chaotic regime) at the intermittent threshold.

One may be tempted to give an alternative definition of bifurcation for smooth dynamical systems in terms of such quantities, the most fundamental of which is the invariant measure, a definition that might still hold for noise-driven systems.

Zeeman has advocated this point of view[31] and proposed to define a noisy bifurcation as a qualitative change in the invariant density. This amounts to introducing a new definition of structural stability where two systems are equivalent if and only if the densities of their invariant measures are differentiably conjugate.

We shall test on specific examples the possible usefulness of that viewpoint. The normal form of the noisy supercritical pitchfork bifurcation reads

$$\dot{x} = \varepsilon x - x^3 + \sigma b(t) \tag{5.1}$$

and the absolutely continuous invariant measure has density

$$\mu(x) = \frac{1}{N} \exp\left[\frac{1}{\sigma^2}\left(\varepsilon x^2 - \frac{x^4}{2}\right)\right]$$

where

$$\begin{aligned}
N &= \left(\frac{|\varepsilon|}{2}\right)^{1/2} \exp\left(\frac{\varepsilon^2}{4\sigma^2}\right) K_{1/4}\left(\frac{\varepsilon^2}{4\sigma^2}\right) && \text{for} \quad \varepsilon < 0 \\
N &= \frac{\sigma^{1/2}\Gamma(1/4)}{2^{3/4}} && \text{for} \quad \varepsilon = 0 \\
N &= \left(\frac{\varepsilon}{2}\right)^{1/2} \exp\left(\frac{\varepsilon^2}{4\sigma^2}\right)\left[K_{1/4}\left(\frac{\varepsilon^2}{4\sigma^2}\right) + \pi 2^{1/2} I_{1/4}\left(\frac{\varepsilon^2}{4\sigma^2}\right)\right] && \text{for} \quad \varepsilon > 0
\end{aligned} \tag{5.2}$$

($K_{1/4}$ and $I_{1/4}$ are modified Bessel functions). Maxima of the density are located at the stable deterministic fixed points and the transition from one peak density to a double-peaked one occurs at the deterministic bifurcation value $\varepsilon = 0$ for any level of noise σ, no noise-induced shift of the bifurcation point being observed (see Fig. 3).

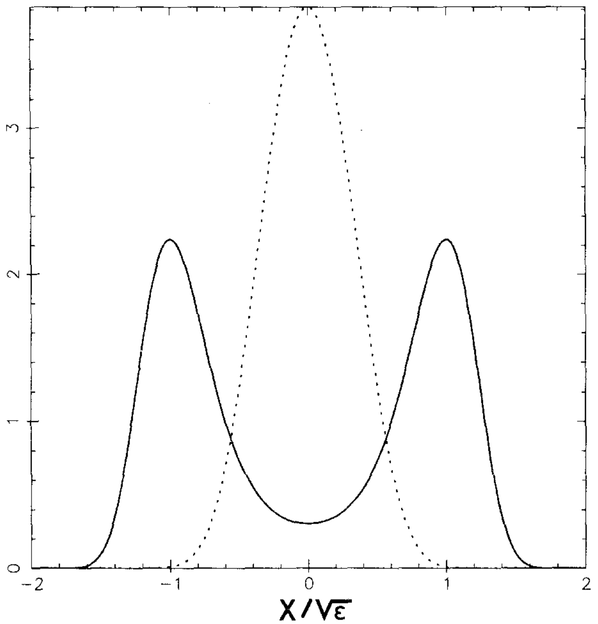

Fig. 3. Invariant density for the noisy pitchfork bifurcation ($\sigma=0.05$): (——) $\varepsilon=0.1$, (···) $\varepsilon=-0.1$.

This behavior does not agree with rough intuition. A double fixed-point structure blurred by noise should behave, for $\varepsilon>0$ and small values of the ratio $\alpha=\varepsilon/\sigma$ (the relevant intrinsic parameter in the problem we consider), barely differently from a unique noisy fixed point. Therefore we expect, *a priori*, a shift of the bifurcation point (assuming that such a point may be defined) toward positive values of ε. Such a shift is observed on other quantities. Consider, for instance, for positive ε, the mean transit time from one of the stable deterministic fixed points ($\sqrt{\varepsilon}$, for instance) to the unstable deterministic fixed point 0,

$$\tau(\varepsilon,\sigma)=\frac{2}{\sigma^2}\int_{-\sqrt{\varepsilon}}^{0} dx \int_{-\infty}^{x} dy \exp\left[\frac{2}{\sigma^2}\int_x^y dz\,(\varepsilon z-z^3)\right] \tag{5.3}$$

As could be expected, this quantity diverges when the level of noise tends to zero. We set $\tau(\varepsilon,\sigma)=T^*(\alpha)/\sigma$, where α is the intrinsic parameter and

$$T^*(\alpha)=2\alpha\int_{-1}^{0} dx \exp\left[\alpha^2\left(\frac{x^4}{2}-x^2\right)\right]\int_{-\infty}^{x} dy \exp\left[\alpha^2\left(y^2-\frac{y^4}{2}\right)\right] \tag{5.4}$$

As α is varied, T^* displays a smooth transition between two distinct regimes (see Fig. 4). For small values of α (the fixed points are then close together and the transition between them is governed by local fluctuations)

$$T^*(\alpha) \sim \alpha^{1/2}\Gamma(1/4)/2^{3/4} \tag{5.5}$$

whereas for large values of α (the fixed points are then widely separated and the transition between them is governed by large deviations) the Laplace formula yields

$$T^*(\alpha) \sim (\pi/2^{1/2}\alpha)\exp(\alpha^2/2) \tag{5.6}$$

We may roughly estimate the location of the transition zone by the vanishing of the second derivative; this yields $\alpha \approx 0.6$. This transition in the range $\varepsilon > 0$ has no conspicuous counterpart in the behavior of the invariant density.

We now consider the saddle-node bifurcation. The noisy normal form is

$$\dot{x} = \varepsilon + x^2 + \sigma b(t) \tag{5.7}$$

No invariant measure exists for this problem. We shall therefore consider related problems with a different behavior of trajectories at infinity.

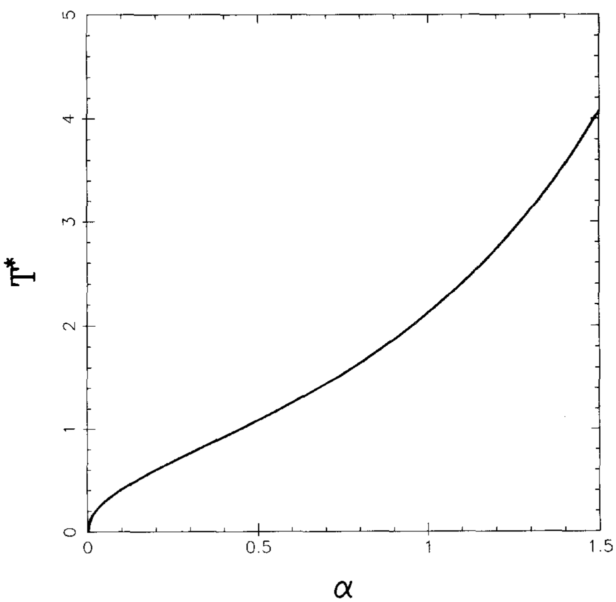

Fig. 4. Mean transit time T^* versus intrinsic parameter α.

First we define a flow on a compactified space by assuming that whenever a trajectory goes to $+\infty$ it reenters at $-\infty$. The density $\mu(x)$ of the invariant probability measure, which satisfies $\mu(\infty)=0$, then is

$$\mu(x)=\frac{1}{N(\sigma,\,\varepsilon)}\exp\left[\frac{2}{\sigma^2}\left(\varepsilon x+\frac{x^3}{3}\right)\right]\int_x^\infty dy\exp\left[-\frac{2}{\sigma^2}\left(\varepsilon y+\frac{y^3}{3}\right)\right] \tag{5.8}$$

where

$$\begin{aligned} N(\sigma,\varepsilon)&=\left(\frac{\pi}{2}\right)^{1/2}\sigma|\varepsilon|^{1/4}\int_0^\infty \frac{dz}{z^{1/2}}\exp\left\{-2|\alpha|^{1/2}\left[\text{sg}(\alpha)+\frac{z^3}{12}\right]\right\} \\ &=\left(\frac{\pi}{2}\right)^{1/2}\sigma|\varepsilon|^{1/4}N(\alpha) \end{aligned} \tag{5.9}$$

α is the intrinsic parameter ε^3/σ^4 and the notation sg(α) stands for sign of α. Using the Laplace formula, one easily checks that, when σ tends to zero, $\mu(x)\,dx$ converges to the deterministic invariant measure. This latter measure is a Dirac measure located at the stable fixed point for $\varepsilon\leqslant 0$ and an absolutely continuous measure with Lorentzian density $\varepsilon^{1/2}/[\pi(\varepsilon+x^2)]$ in the opposite case.

For a given level of noise σ, the invariant density profile displays no qualitative change when ε is varied (see Fig. 5) other than a shift of the maximum toward negative values of ε (for $\varepsilon>0$ or slightly negative) or toward positive values of ε (for $\varepsilon<0$ and $|\alpha|$ large enough).

To estimate this shift, we may calculate the mean value of x, which, in the deterministic case, is zero for positive values of ε (we actually misuse a bit the expression "first moment," since the corresponding integral tends only symmetrically to zero) and to $-|\varepsilon|^{1/2}$ for negative values of ε. This yields

$$\langle x\rangle=-\frac{|\varepsilon|^{1/2}}{2}\frac{1}{N(\alpha)}\int_0^\infty dy\,y^{1/2}\exp\left\{-2|\alpha|^{1/2}\left[\text{sg}(\alpha)\,y+\frac{y^3}{12}\right]\right\} \tag{5.10}$$

For $\varepsilon<0$ the shift obviously originates in the deterministic dynamics. The system reads

$$\dot{y}=-2(-\varepsilon)^{1/2}y+y^2+\sigma b(t) \tag{5.11}$$

where $y=x+|\varepsilon|^{1/2}$ is the departure from the stable deterministic fixed point $-|\varepsilon|^{1/2}$. The effect of the nonlinear deterministic term adds to noise-induced deviations toward positive y values and counteracts deviations in the opposite direction. Such a reasoning is valid only when the nonlinear term does not play a dominant part in the dynamics near the deterministic

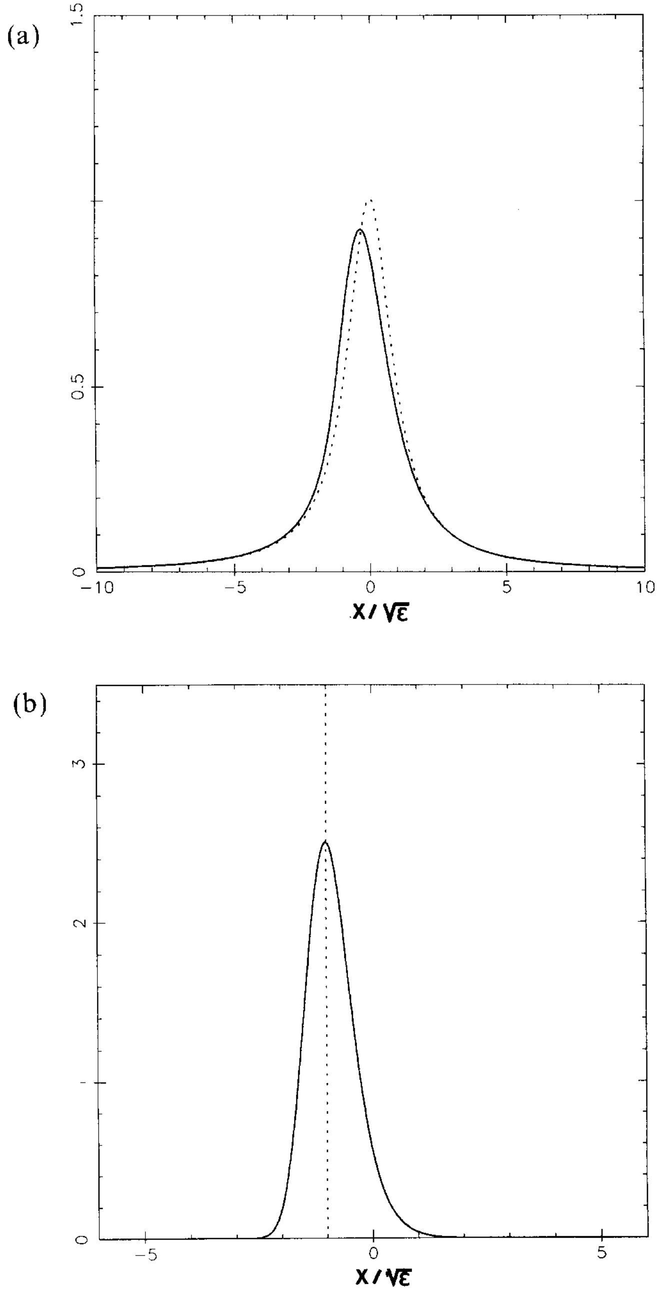

Fig. 5. Invariant density for the noisy saddle-node bifurcation, assuming cyclical reentrance: (a) (···) $\varepsilon = 0.1$, $\sigma = 0$ (deterministic case); (——) $\varepsilon = 0.1$, $\sigma = 0.178$ (noisy case). (b) (···) $\varepsilon = -0.1$, $\sigma = 0$ (deterministic case); (——) $\varepsilon = -0.1$, $\sigma = 0.178$ (noisy case, $\alpha = 1$).

fixed point, that is, for $|\alpha|$ large enough, and yields a simple explanation for the observed shift toward positive values of x.

Higher order moments of the invariant measure cannot be defined. In fact, the asymptotic form for high values of $|x|$ is Lorentzian and reads

$$\frac{1}{x^2}\left[\left(\frac{|\varepsilon|}{2\pi}\right)^{1/2}\frac{1}{|\alpha|^{1/4}N(\alpha)}\right] \tag{5.12}$$

In contrast, higher moments behave very differently in the deterministic case, according to the sign of ε; for $\varepsilon>0$ the measure is Lorentzian and no moment of order higher than one exists, whereas in the opposite case all moments are, loosely speaking, zero.

Thus, for finite noise level σ, the invariant density displays no qualitative change in profile or moments when ε is varied except for a first kind discontinuity at $\varepsilon=0$ in the derivative $d\langle x\rangle/d\varepsilon$ of the first moment.

In the intermittent transition to chaos, trajectories are reinjected randomly inside the finite region where the saddle-node bifurcation takes place.[32] The behavior of the invariant mesure is nonetheless similar to the previous example. In the deterministic case, the invariant mesure undergoes at the bifurcation point $\varepsilon=0$ a weakly continuous transition from Dirac measure to an absolutely continuous and locally Lorentzian measure.[33,34] In the presence of noise, the measure is always continuous, with a sharp peak in the region where the saddle-node bifurcation takes place and its density displays no transition.

These examples show that a noisy bifurcation is not necessarily connected with a qualitative change in the profile of the invariant density; they also suggest that such changes—*if any*—occur at the deterministic bifurcation value. Thus, the invariant measure, though a quantity of major interest, seems inadequate for investigating the detailed effects of noise on bifurcations, especially if the study is restricted to the sole profile and its qualitative changes. It is therefore necessary to examine other relevant quantities.

6. LIAPUNOV EXPONENT

Deterministic systems exhibit a singularity in the Liapunov exponent at a bifurcation value.

First we consider the supercritical pitchfork bifurcation. In the deterministic case, the Liapunov exponent is then given by the derivative of the vector field at the stable fixed point: for $\varepsilon<0$, $\lambda(\varepsilon)=\varepsilon$, whereas for $\varepsilon>0$, $\lambda(\varepsilon)=-2\varepsilon$ (see Fig. 6). The derivative $d\lambda/d\varepsilon$ is thus discontinuous at $\varepsilon=0$

(where the exponent vanishes). When noise is added, the Liapunov exponent then reads

$$\begin{aligned}
\lambda &= \frac{\varepsilon}{2}\left[3\,\frac{K_{3/4}(\alpha^2/4)}{K_{1/4}(\alpha^2/4)} - 1\right] && \text{for} \quad \varepsilon < 0 \\
\lambda &= -\frac{6\pi}{\Gamma(1/4)^2}\,\sigma && \text{for} \quad \varepsilon = 0 \qquad (6.1) \\
\lambda &= -\frac{\varepsilon}{2}\left[3\,\frac{I_{3/4}(\alpha^2/4) + I_{-3/4}(\alpha^2/4)}{I_{1/4}(\alpha^2/4) + I_{-1/4}(\alpha^2/4)} + 1\right] && \text{for} \quad \varepsilon > 0
\end{aligned}$$

where I and K are modified Bessel functions and $\alpha = \varepsilon/\sigma$ is the intrinsic parameter. For small values of α we obtain quite logically the asymptotic expression

$$\lambda \sim -\frac{6\pi}{\Gamma(1/4)^2}\,\sigma \qquad (6.2)$$

When ε is varied, two distinct regimes may still be observed (see Fig. 6). However, the transition has been smoothened by noise and a quadratic

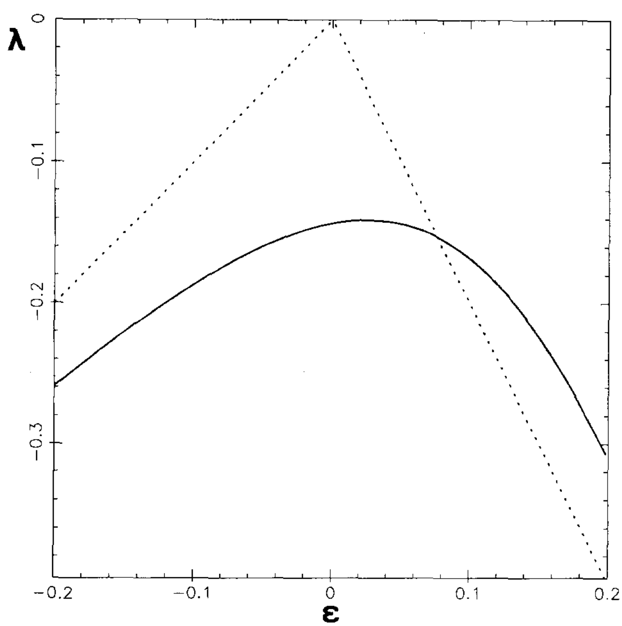

Fig. 6. Liapunov exponent for the noisy supercritical pitchfork bifurcation versus bifurcation parameter: $(\cdots)\sigma = 0$ (deterministic case), (——) $\sigma = 0.1$ (noisy case).

maximum at $\alpha \approx 0.25$ replaces the previous singularity. One notices that the Liapunov exponent no longer vanishes ($\lambda \approx -1.4\sigma$ at its maximum); it is increased with respect to its deterministic value for high values of ε $\{\lambda = -2\varepsilon[1 - 3/4\alpha^2 + O(1/\alpha^4)]$ for $\varepsilon > 0$ and high values of $\alpha\}$ and decreased for low values of ε $\{\lambda = \varepsilon[1 + 3/\alpha^2 + O(1/\alpha^4)]$ for $\varepsilon < 0$ and high values of $\alpha\}$. Noise thus affects the system's stability. Moreover, the maximum of the exponent is shifted toward positive values of ε, as could be expected on intuitive grounds. We also remark that the transition occurs at distinct values of α for the Liapunov exponent and the mean transit time (cf. Section 5). This stems from the fact that these two quantities are basically different. The mean transit time is a probabilistic quantity, meaningless in the deterministic case, and measures noise-induced deviations. On the other hand, the Liapunov exponent is a dynamical quantity, well defined for the deterministic system, which smoothly varies with noise intensity and is connected with correlation properties.

We now examine the saddle-node bifurcation.

If we impose no condition on the long-distance behavior of trajectories, the Liapunov exponent is undefined for $\varepsilon > 0$, since solutions blow up in a finite time.

We therefore rather consider the related well-behaved systems we introduced in Section 5.

In the case of cyclical reentrance at $-\infty$, we have $\lambda = 2\langle x \rangle$, where $\langle x \rangle$ is given by formula (5.10). For the deterministic systems, λ is equal to $-2|\varepsilon|^{1/2}$ for $\varepsilon < 0$ and is zero for $\varepsilon > 0$ [assuming that, though the integral (2.21) is only symmetrically convergent, we may still define in some sense λ]. When noise is added to the system, the Liapunov exponent is well-defined for $\varepsilon > 0$. It is nonpositive and steadily increases with ε, no longer displaying any singularity at the deterministic bifurcation value $\varepsilon = 0$ (cf. Section 5). It is shifted toward lower values for $\varepsilon > 0$ or slightly negative and toward higher values in the remainder of the parameter range.

We now examine a simple model of intermittent transition: We assume that whenever a trajectory leaves the interval $[-\Delta, \Delta]$, where its dynamics is given by the saddle-node normal form (5.7), it immediately reenters this region with uniform probability. The Liapunov exponent λ vanishes at $\varepsilon = 0$ with a square root singularity. When noise is added, λ can be computed numerically (see Fig. 7). The singularity then disappears: The derivative $d\lambda/d\varepsilon$ remains finite at the point $\varepsilon^* > 0$ where λ vanishes. It is tempting to consider ε^* as the new bifurcation value, as the transition from periodic to aperiodic regime actually occurs at this point. This may be quite justified from an operational viewpoint, since the Liapunov exponent is the relevant "order parameter" in transitions from order to chaos. However, one should probably not jump to conclusions, since other quantities of interest will certainly display transitions between the two regimes at other values of ε. One also notices that the system's stability is decreased with respect to its deterministic value noise for $\varepsilon > 0$ and increased for sufficiently negative values of ε.

We close the present discussion with two remarks:

1. The Liapunov exponent is rather insensitive to the Gaussian nature of noise: Quasi-identical results are obtained for Gaussian and uniform noises of the same variance.

2. Similar results were obtained by Hirsch *et al.*[35] They observe a shift of the bifurcation point toward *negative* values of ε. This difference from our results arises from the adoption of a different reentrance process in their paper. Indeed, though the analytic part of their paper deals with a uniform reentrance process, the numerical computation of the Liapunov exponent is performed using the logistic map. This illustrates the high sensitivity of the Liapunov exponent to this feature of the model. This point is further illustrated by the opposite effects of noise on stability

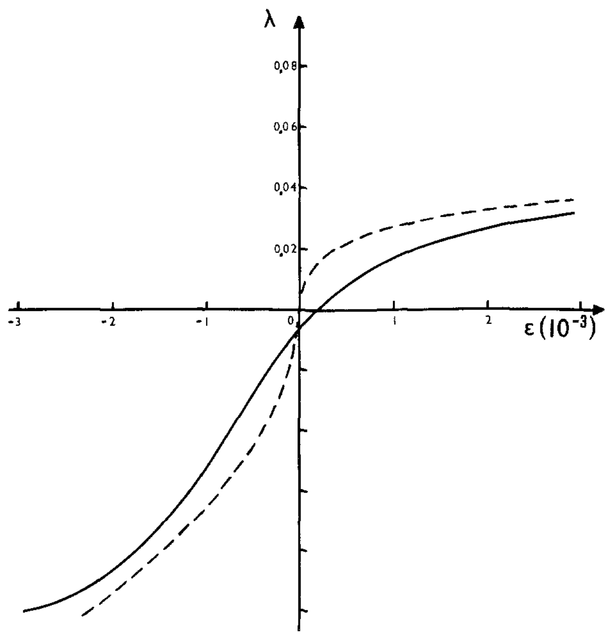

Fig. 7. Liapunov exponent for intermittency versus bifurcation parameter: (– –) $\sigma = 0$ (deterministic case), (——) $\sigma = 0.02$ (noisy case). Uniform reentrance in the interval $[-0.08, 0.08]$ is assumed. Quasi-identical results are obtained for Gaussian and uniform noises with the same variance.

displayed by the two models we analyzed (cyclical reentrance versus random reentrance).

Thus, unlike the invariant measure, the Liapunov exponent displays noise-induced shifts of the transition point between qualitatively different regimes. However, this transition in most cases is perfectly smooth. Moreover, if we consider other quantities (invariant measure, exit times; cf. Section 5), we observe transitions for distinct values of the control parameter. This forbids us to define an effective bifurcation value. In the following sections, we focus on the more appropriate notion of "bifurcation region."

7. EFFECTIVE POTENTIAL

Limiting our study to the pitchfork bifurcation, we now investigate the properties of the effective potential $V_{\varepsilon,\sigma}$ (cf. Section 2). This potential will allow us to define an effective bifurcation parameter $\varepsilon_r(\varepsilon, \sigma)$ and an extended "bifurcation zone." The generating functional of correlation functions (2.28) reads

$$Z[J] = \frac{1}{Z} \int \mathscr{D}x \exp\left(i \int dt\, Jx\right) \exp\left\{-\frac{1}{\sigma^2} \times \int dt \left(\frac{\dot{x}^2}{2} + \frac{\varepsilon^2}{2} x^2 - \varepsilon x^4 + \frac{x^6}{2} - \frac{\sigma^2}{2} x^2\right)\right\} \tag{7.1}$$

In the noiseless limit $\sigma^2 \to 0$ the associated effective potential simply reads

$$V_{\varepsilon,\sigma}(\phi) = \tfrac{1}{2}\varepsilon^2\phi^2 - \varepsilon\phi^4 + \tfrac{1}{2}\phi^6 \tag{7.2}$$

Neglecting nonlinear terms, the two-point Green's function (2.29) is given by

$$G^{(0)}(t, t') = = (\sigma^2/2\varepsilon) \exp(-\varepsilon |t - t'|) \tag{7.3}$$

We define the effective (renormalized) bifurcation parameter by

$$\varepsilon_r^2(\varepsilon, \sigma) = (d^2 V_{\varepsilon,\sigma}/d\phi^2)_{\phi=0} \tag{7.4}$$

It reduces in this limit to $\varepsilon_r = \varepsilon$. Whereas for $\varepsilon < 0$, $V_{\varepsilon,0}$ has only one quadratic minimum, when ε becomes positive, it develops three minima related to the deterministic fixed points (see Fig. 8). Therefore, the deterministic bifurcation point $\varepsilon = 0$ corresponds to a change in the form of the "noiseless" effective potential $V_{\varepsilon,0}$.

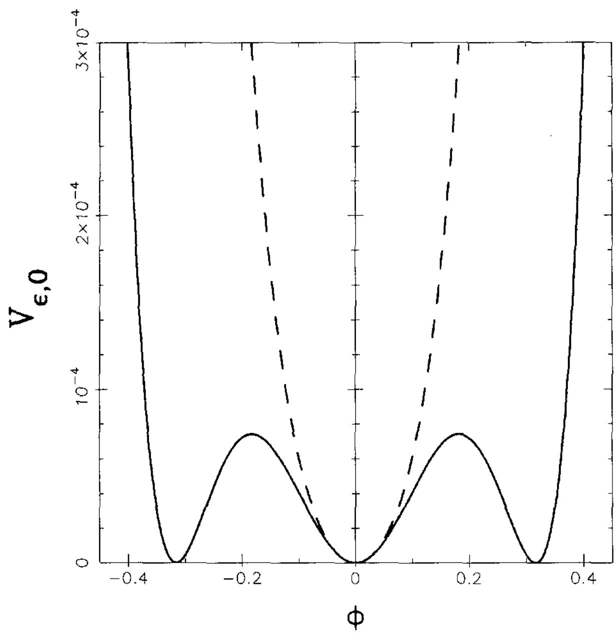

Fig. 8. Lowest order approximation $V_{\varepsilon,0}$ to the effective potential: (—) $\varepsilon = 0.1$, (– –) $\varepsilon = -0.1$.

Now we consider the qualitative effects of a small noise on the deterministic system. We expand the potential in powers of σ and restrict outselves to the first corrective term $-3\sigma^2\phi^2/2$ to the noiseless effective potential. This corrective term entails qualitative changes in the potential profile. A region arises, delimited by $|\varepsilon| < 3^{1/2}\sigma$, where the effective potential develops two neighboring minima (see Fig. 9). This region, which obviously dissapears at $\sigma = 0$, separates the one-fixed-point zone ($\varepsilon < -3^{1/2}\sigma$) from the three-fixed-point zone ($\varepsilon > 3^{1/2}\sigma$). Notice that in the three-fixed-point region the external minima

$$x_\pm = \pm\varepsilon^{1/2}\left[1 + \frac{3}{4\alpha^2} + O\left(\frac{1}{\alpha^4}\right)\right] \tag{7.5}$$

are slightly shifted with respect to the deterministic stable fixed points $\pm\varepsilon^{1/2}$ (the explanation is the same as in Section 5), whereas in the two-well zone the two minima are not located near those fixed points.

One may interpret the appearance of such an extended region as a blurring by noise of the bifurcation point. It is related to the fact that the different relevant quantities in the problem are regular functions of the bifurcation parameter and, since the local or global character in space and

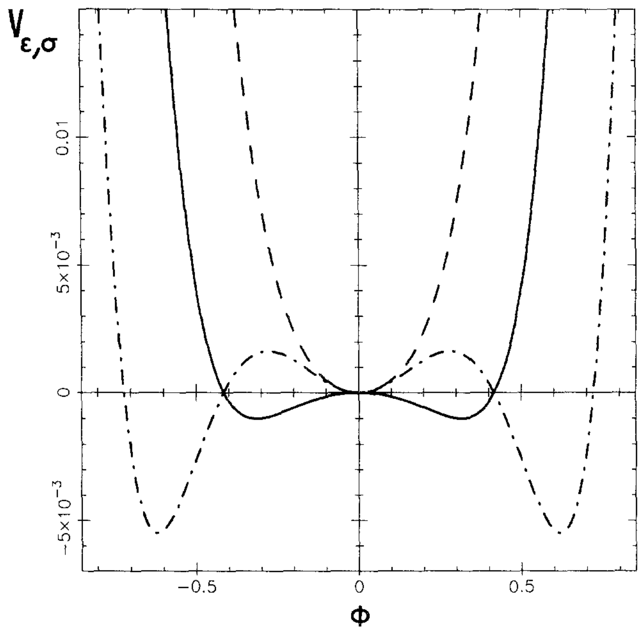

Fig. 9. Effective potential $V_{\varepsilon,\sigma}$ to first order: (-·-) $\sigma = 0.1$, $\varepsilon = 2\sqrt{3}\sigma$; (——) $\sigma = 0.1$, $\varepsilon = 0$; (- -) $\sigma = 0.1$, $\varepsilon = -2\sqrt{3}\sigma$.

the characteristic dynamical time scale depend on the quantity that is considered, the qualitative changes of regimes they undergo occur for distinct values of this parameter. Thus, in the presence of noise, the effective potential displays a "bifurcation region," rather than a bifurcation point. The boundaries of that region are given by

$$\varepsilon_r(\varepsilon, \sigma) = 0 \tag{7.6}$$

For arbitrary σ, we shall define the transition region by the same criterion.

In principle, the effects of noise on the effective potential can be rigorously studied using perturbative expansions, such as the loop expansion of $Z[J]$. For instance, one may think of expanding it in powers of σ, the first term being no other than the previously studied "noiseless" effective potential. This expansion is not trivial, since the action itself depends on σ and a consistent ordering thus becomes difficult. Moreover, we remark that the noiseless limit, which corresponds to mean field theory, becomes meaningless near the deterministic bifurcation point $\varepsilon = 0$ due to the non-Gaussian character of fluctuations. Therefore, a small-σ expansion is meaningless in the bifurcation region and we shall proceed differently.

Since, as shown before, the relevant quantities mainly depend on the intrinsic parameter $\alpha = \varepsilon/\sigma$, it is necessary to introduce a scaling transformation to take into account the essential effects of noise in the bifurcation region. After the change of scale

$$x \to x/\sigma^{1/2}, \qquad t \to \sigma t, \qquad J \to \sigma^{5/2} J \tag{7.7}$$

the generating functional may be written

$$Z[J] = \frac{1}{Z} \int Dx \exp\left(i \int dt\, Jx\right) \exp\left[-\int dt \left(\frac{\dot{x}^2}{2} + \frac{\alpha^2}{2} x^2 - \alpha x^4 + \frac{x^6}{2} - \frac{3}{2} x^2\right)\right] \tag{7.8}$$

In terms of a graphical representation, the loop expansion of the effective potential contains a subclass of diagrams with self-lines (obtained by pairing two legs at a given vertex). These "infrared"-divergent terms can be summed over by introducing the Wick ordering of monomials, denoted by $:x^n:$, in the action. We take the rescaled version of (7.3) as the free propagator to compute the Wick monomials, thus considering as usual the negative "mass" term as part of the interaction potential.[14] After Wick ordering we finally obtain

$$Z[J] = \frac{1}{Z} \int Dx \exp\left(i \int dt\, Jx\right) \exp\left\{-\int dt \left[\frac{\dot{x}^2}{2} + \frac{m^2 :x^2:}{2} + \left(\frac{15}{4} - \alpha\right) :x^4: + \frac{:x^6:}{2}\right]\right\} \tag{7.9}$$

where $m^2 = 45/4\alpha^2 - 9 + \alpha^2$. The zero-loop effective potential is now

$$V_\alpha^{(0)}(\phi) = \frac{m^2}{2} \phi^2 + \left(\frac{15}{4} - \alpha\right) \phi^4 + \frac{\phi^6}{2} \tag{7.10}$$

The contribution of the following term in the perturbation series comes from the two-loops diagram:

$$V_\alpha^{(2)}(\phi) = -3(10\phi^2 + 15 - 4\alpha)^2\, \phi^2 I(\alpha) \tag{7.11}$$

where $I(\alpha)$ is given by

$$I(\alpha) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{dk_1\, dk_2}{[(k_1 + k_2)^2 + m^2](k_1^2 + m^2)(k_2^2 + m^2)} \tag{7.12}$$

Using the identity $1/a = \int_0^\infty dp\, e^{-ap}$, we can transform the integral into

$$\frac{1}{4\pi}\int_0^\infty \frac{dp_1\, dp_2}{p_1 + p_2}\int_0^\infty dp_3 \frac{\exp(-m^2 p_3)}{p_1 p_2/(p_1+p_2) + p_3}$$

$$= -\frac{1}{4\pi}\int_0^\infty \frac{dp_1\, dp_2}{p_1 + p_2} \exp\left(m^2 \frac{p_1 p_2}{p_1 + p_2}\right) \exp[-m^2(p_1 + p_2)$$

$$\times \mathrm{Ei}\left(-m^2 \frac{p_1 p_2}{p_1 + p_2}\right) \tag{7.13}$$

where Ei denotes the exponential integral function.[36] The change of variables $p_1 + p_2 = u$, $p_1 = uv$ then yields

$$I(\alpha) = \frac{1}{4\pi m^2}\int_0^{1/2} dv \ln\left(1 + \frac{1}{v - v^2}\right) = \frac{0.0856}{m^2} \tag{7.14}$$

Therefore, to second order in the loop expansion, the renormalized bifurcation parameter is given by

$$\varepsilon_r(\alpha) = m^2 - \frac{8.22}{m^2}\left(\frac{15}{4} - \alpha\right)^2 \tag{7.15}$$

which yields for the bifurcation region the estimate $\alpha = (-0.73, 0.82)$.

8. CONCLUSION

We now summarize the results of our study and give an elementary explanation for the appearance of an extended bifurcation region in noisy dynamical systems.

Let us list our main conclusions:

1. In noisy systems topological concepts become meaningless. Bifurcations must be redefined in terms of invariant measure, dynamical quantities, and probabilistic ones.

2. Noise smoothens the transition: It suppresses singularities (as observed in every model we consider).

3. Dynamical or probabilistic quantities generally display noise-induced shifts of the transition point between the two regimes.

4. Such a shift never occurs for the invariant measure and its moments. The changes, if any, occur at the deterministic bifurcation value.

5. Since the different relevant quantities undergo transitions for distinct values of the control parameter, it is meaningless to define an effective bifurcation value.

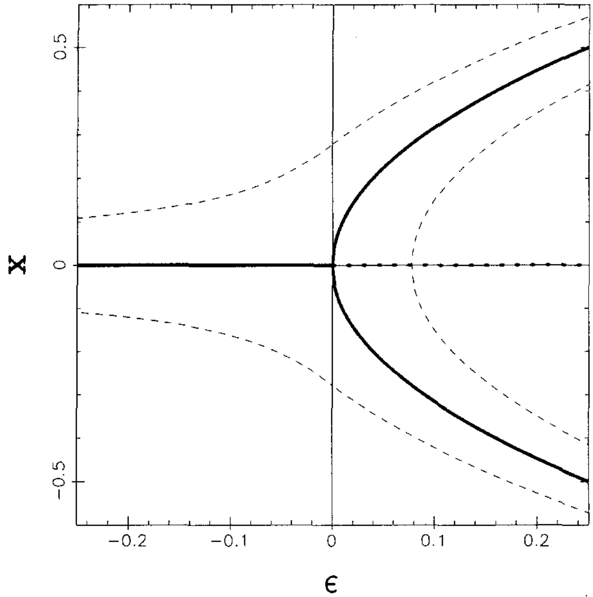

Fig. 11. Noisy bifurcation diagram for supercritical pitchfork bifurcation displaying (——) stable fixed points and (···) unstable ones. Stable fixed points are surrounded by a zone of local fluctuations (area enclosed by the dashed curves). The region of non-Gaussian fluctuations approximately ranges from $\varepsilon = -\sigma$ to $\varepsilon = \sigma$. Here $\sigma = 0.1$.

6. One may introduce the more appropriate concept of a bifurcation region inside which the transitions discussed above take place. In fact, such a region naturally appears when one studies quantities such as the effective potential.

Restricting ourselves for the sakes of clarity to the pitchfork bifurcation, we now give a simple interpretation of the bifurcation region.

Figure 10 pictures the "noisy" bifurcation diagram. Each stable fixed point is surrounded by a zone which depicts the blurring due to local fluctuations. We have defined this region as the set of points where the invariant density exceeds half its maximum value. For $|\varepsilon|$ large enough, each fixed point is embedded in a zone, the width of which is of order $\sigma/\sqrt{\varepsilon}$ (one or two nearly Gaussian peaks in the invariant density). The situation changes near $\varepsilon = 0$, where local fluctuations are no longer Gaussian and the standard deviation is on the order of $\sigma^{1/2}$. We are then faced with a more complex structure (two neighboring deterministic stable fixed points separated by an unstable one), which may be assimilated to a single "noisy" feature.

This is analogous to what we observed in the effective potential (cf. Section 7). In fact, let us discuss the quantum problem associated with this potential.

For ε sufficiently negative, the potential presents a single well corresponding to the deterministic stable fixed point (cf. Fig. 9); the ground state has one single maximum and the dynamics is characterized by the single time scale $1/\varepsilon$ corresponding to motion in the well.

For ε positive and large enough, the potential exhibits two deep wells corresponding to the stable fixed points. They are separated by a metastable state associated with the unstable fixed point. The ground state is localized in the two external wells and the dynamics exhibits two distinct time scales: the rapid scale of motion in a well (local fluctuations) and the slow time scale of tunneling (transit between the two fixed points due to large deviations).

In the intermediate range of ε, the potential has two wells. The ground state is not localized in these wells and a single time scale governs the dynamics, as in the single-fixed-point case.

We may therefore define the bifurcation region as the range of parameter values where nonlinear terms play a prominent part in the dynamics and fluctuations are not Gaussian. Since the standard deviation for such fluctuations is of order σ, this yields the estimate $|\varepsilon| \leqslant \sigma$ for the bifurcation region, in agreement with the results of Section 7. This agreement is not fortuitous, as we define the endpoints of the bifurcation region in the previous section by the vanishing of the "mass term" in the effective potential. It corresponds precisely to the onset of non-Gaussian fluctuations.

The upper bound (whose value is only indicative, due to the smoothness of the transition) corresponds to the merging of the two "dressed" fixed points. It may be estimated as the value of ε at which equality of the two time scales (local fluctuations and large deviations) occurs. It may also be defined as the value at which the two stable fixed points are separated by about twice the standard deviation $\sigma\varepsilon^{-1/2}$.

As the deterministic system has a single fixed point in the range $\varepsilon \leqslant 0$, we could not think of any definition of the lower bound other than the previous one, which is based on the nature of fluctuations.

Finally, it is of note that the signature of the noisy bifurcation on the invariant density resides not so much in the appearance of a double peak at $\varepsilon = 0$ as in the existence of an extended region around this value where local fluctuations are not Gaussian.

ACKNOWLEDGMENTS

We express our deep gratitude to Drs. P. Collet, J.-F. Luciani, and R. Seneor for many fruitful discussions.

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